Gravitation and Linear Transformation Paradoxes

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Abstract Two examples are presented showing that gravitation does not transform by linear transformations.

Résumé Nous présentons deux exemples démontrant qu’il n’est pas possible de transformer la gravitation par transformation linéaire.

Key words: Lorentz transformation, energy-momentum tensor
Introduction

Gravitation is assumed to be generally covariant. It would then be covariant with respect to linear transformations.

1 Lorentz Transformation

Let $\mathcal{F}_v'$ and $\mathcal{F}_v$ be frames of reference with coordinates $x', t'$ and $x, t$ respectively related by the Lorentz transformation with $c = 1$

$$x' = \frac{x - vt}{\sqrt{1 - v^2}} \quad y' = y \quad z' = z \quad t' = \frac{t - vx}{\sqrt{1 - v^2}}$$

(1)

With respect to $\mathcal{F}_v'$ let there be a stationary particle $A$ at the origin and let $M'$ be the total energy of the system. With respect to $\mathcal{F}_v'$ we have in harmonic coordinates with $R'^2 \equiv x'^2$ that [1]

$$d\tau^2 = \frac{1 - M'G/R'}{1 + M'G/R'} dt'^2 - \frac{1 + M'G/R'}{1 - M'G/R'} \frac{M'^2G^2}{R'^4} (x' \cdot d\mathbf{x}')^2$$

(2)

Now $g'_{01} = 0$ so by (1)

$$g_{00}(x, t) = \frac{\partial x'^\alpha}{\partial x'^0} \frac{\partial x'^\beta}{\partial x^0} g'_{\alpha\beta} = \frac{g'_{00}(\frac{x-vt}{\sqrt{1-v^2}}, y, z) + v^2 g'_{11}(\frac{x-vt}{\sqrt{1-v^2}}, y, z)}{1-v^2}$$

(3)

For each $\mathcal{F}_v'$ a total energy $M' = \sqrt{1-v^2}m$ is chosen. As $v \to 1$ we have $M' \to 0$ so the rest mass of $A$ approaches zero but the total energy with respect to $\mathcal{F}_v$ is $m$. By (2) and (3) as $v \to 1$

$$g_{00}(x, t) \to -1 + \frac{4Gm}{|x-t|}$$

(4)

The limiting gravitational field will have no energy. Consider a particle $B$ of zero rest mass moving along the $x$ axis and at infinity with momentum in the direction of the negative $x$ axis. There is no gravitational field of $B$ in advance of $B$. The speed of $B$ is determined by [1]

$$g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = 0$$

(5)
Using the limiting values of $g_{00}, g_{01}$ and $g_{11}$ we have by (5)

$$\frac{dx}{dt} = \frac{4Gm - |x-t|}{4Gm + |x-t|} \quad \frac{dy}{dt} = \frac{dz}{dt} = 0$$

Consequently $B$ will not reach a point with $x = t$ contradicting the conservation of momentum.

## 2 Galilean Transformation

Let $g_{\mu\nu}(x)$ and $g'_{\mu\nu}(x)$ be two solutions of the Einstein field equations for a given energy-momentum tensor. The field equations and Bianchi identities imply [1]

$$T^\mu{}_{\nu\rho} = \frac{\partial T^\mu{}_{\nu\rho}}{\partial x^\rho} + \Gamma^\mu{}_{\nu\alpha} T^\alpha{}_{\rho} + \Gamma^\mu{}_{\rho\alpha} T^\alpha{}_{\nu} = 0 \quad \frac{\partial T^\mu{}_{\nu}}{\partial x^\rho} + \Gamma^\mu{}_{\nu\alpha} T^\alpha{}_{\rho} + \Gamma^\mu{}_{\rho\alpha} T^\alpha{}_{\nu} = 0$$

where $\Gamma^\mu_{\nu\alpha}$ is formed from $\Gamma^\mu_{\nu\alpha}$ by replacing $g_{\mu\nu}(x)$ with $g'_{\mu\nu}(x)$. Consider a system with $T^{\alpha\beta}(x) = 0$ for $\alpha + \beta \neq 0$ and a solution $g_{\mu\nu}$ such that

$$\frac{\partial g_{\mu\nu}}{\partial x^\rho} = 0 \quad g_{01} = g_{02} = g_{03} = 0$$

We have by (7) with $\mu = 0$ at points where $T^{00}$ is not zero that

$$\Gamma^0_{0\rho} + \Gamma^\rho_{00} = \Gamma^0_{0\rho} + \Gamma^\rho_{00}$$

Now if $g_{\mu\nu}(x)$ is a solution of the Einstein field equations then so is $[1]

$$g'_{\mu\nu}(x) = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$$

Consider the Galilean transformation

$$x'^{0} = x^{0} \quad x'^{1} = x^{1} - vx^{0} \quad x'^{2} = x^{2} \quad x'^{3} = x^{3}$$

By (8), (9), (10) and (11) we have

$$\frac{1}{2} v g^{00} \left[ \frac{\partial g_{00}}{\partial x^1} + v^2 \frac{\partial g_{11}}{\partial x^1} \right] = 0$$

We can let $v = 1$ and $v = 2$ to conclude at points where $T^{00}$ is not zero that

$$\frac{\partial g_{00}}{\partial x^1} = \frac{\partial g_{11}}{\partial x^1} = 0$$

which doesn’t hold in general.
Conclusion

These two examples show that gravitation is not generally covariant.

References