Solutions of Einstein field equations that are not coordinate related.

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Abstract

We show for a system with a nonzero energy-momentum tensor there exists solutions that are not coordinate related. We also show this for a system having only an electromagnetic energy-momentum tensor.

1 Introduction

A solution of the Einstein field equations is determined up to four functional degrees of freedom. This corresponds to the fact if $g_{\mu\nu}$ is a solution then so is $g'_{\mu\nu}$ where $g'_{\mu\nu}$ related to $g_{\mu\nu}$ by a general coordinate transformation $x \rightarrow x'$ [1]. We show for system with nonzero energy-momentum tensor this does not hold. Using a different argument we also show this does not hold for a system with only an electromagnetic energy-momentum tensor. This will mean there are solutions of the Einstein field equations that do not differ by a coordinate transformation.

2 Nonzero energy-momentum tensor

This section generalizes section 2 of [2]. Let $g'_{\mu\nu}(x)$ and $g_{\mu\nu}(x)$ be two solutions of the Einstein field equations for a energy-momentum tensor $T^{\mu\nu}(x)$. The Einstein field equations $G'_{\mu\nu} = 8\pi T^{\mu\nu}$ and $G_{\mu\nu} = 8\pi T^{\mu\nu}$ where $G'_{\mu\nu}$ is formed from $G_{\mu\nu}$ by replacing $g_{\mu\nu}(x)$ with $g'_{\mu\nu}(x)$ imply the conservation of energy and momentum equations [1]

$$\partial_{\alpha} T^{\mu\alpha} + \Gamma^{\mu}_{\alpha\beta} T^{\alpha\beta} + \Gamma^{\alpha}_{\alpha\beta} T^{\mu\beta} = 0 \quad \partial_{\alpha} T^{\mu\alpha} + \Gamma^{\mu}_{\alpha\beta} T^{\alpha\beta} + \Gamma^{\alpha}_{\alpha\beta} T^{\mu\beta} = 0$$ (1)
where $\Gamma^\alpha_{\alpha\beta}'$ is formed from $\Gamma^\lambda_{\alpha\beta}$ by replacing $g_{\mu\nu}(x)$ with $g'_{\mu\nu}(x)$. Setting $\mu = 0$ and subtracting the two equations of (1) gives

$$(\Gamma^0_{\alpha\beta} - \Gamma^0_{\alpha\beta})T^{\alpha\beta} + (\Gamma^\alpha_{\alpha\beta} - \Gamma^\alpha_{\alpha\beta})T^{0\beta} = 0$$

The Einstein field equations have four degrees of freedom corresponding to the fact that if $g_{\mu\nu}(x)$ is a solution of Einstein’s field equations then so is

$$g'_{\mu\nu}(x) \equiv \frac{\partial x^\alpha}{\partial x'\mu}(x)\frac{\partial x^\beta}{\partial x'\nu}(x)g_{\alpha\beta}(x)$$

where $x \rightarrow x'$ is a general coordinate transformation [1]. Consider the Galilean transformation

$$x'^0 = x^0 \quad x'^1 = x^1 - vx^0 \quad x'^2 = x^2 \quad x'^3 = x^3$$

By (3) and (4)

$$g'_{00} = g_{00} + 2g_{01} + v^2g_{11} \quad g'_{01} = g_{01} + vg_{11} \quad g'_{11} = g_{11}$$

$$g'_{02} = g_{02} + vg_{12} \quad g'_{12} = g_{12} \quad g'_{03} = g_{03} + vg_{13} \quad g'_{13} = g_{13} \quad g'_{22} = g_{22} \quad g'_{23} = g_{23} \quad g'_{33} = g_{33}$$

so by (2) and (5)-(7)

$$\frac{1}{2}[Av + Bv^2 + g^{00}(\partial_1g_{11})T^{00}v^3] = 0$$

where $A$ and $B$ are made up of sums of functions of form $kg^{\alpha\beta}(\partial_\gamma g_{\sigma\tau})T^{\mu\nu}$ where $k$ is an integer. Now (8) holds for all $v$ so we must have

$$g^{00}(\partial_1g_{11})T^{00} = 0$$

Begin with a system with metric $g_{\mu\nu}$ satisfying the Einstein field equations with energy-momentum tensor $T^{\mu\nu}$ and $g^{00}(\partial_1g_{11})T^{00} \neq 0$. We would then have that the metric $g'_{\mu\nu}$ given by (5)-(7) does not satisfy the Einstein field equations with energy-momentum tensor $T^{\mu\nu}$. There are four degrees of freedom in the choice of the coordinate transformation $x \rightarrow x'$ in (3) and the Einstein field equations have four degrees of freedom. We have just shown that not all metrics related to $g_{\mu\nu}$ as in (3) are solutions of the Einstein field equations with energy-momentum tensor $T^{\mu\nu}$. Consequently we must have solutions of the Einstein field equations with energy-momentum tensor $T^{\mu\nu}$ that do not differ by a coordinate transformation.
3 Electromagnetic energy-momentum tensor

This section is a revision of [3]. Let $A_\mu(x)$, $g_{\mu\nu}(x)$, and $A^\mu(x) = g^{\nu\alpha}(x)A_\alpha(x)$ be the component functions of the electromagnetic potential, metric tensor, and electromagnetic vector potential respectively where $x$ is a point of $\mathbb{R}^4$.

Require the component functions to be smooth functions. The electromagnetic field is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(10)

A gauge transformation

$$A_\mu \rightarrow A_\mu + \partial_\mu \phi$$

(11)

where $\phi(x)$ is a smooth function on $\mathbb{R}^4$ leaves $F_{\mu\nu}$ unchanged. By (11) a gauge transformation transforms the electromagnetic vector potential as

$$A^\mu \rightarrow \tilde{A}^\mu \equiv A^\mu + g^{\mu\alpha}\partial_\alpha \phi$$

(12)

Require of the metric and electromagnetic vector potential that there are real numbers $a$ and $b$ such that $g^{00}(x) < -a < 0$ and $|A^0(x)| < b$ for all $x$. Letting $\phi(x) = -(1+b)x^0/a$ we have by (12) that $\tilde{A}^0(x) > 1$ for all $x$. There are then coordinates $x'$ such that $\tilde{A}^0 = 1$ and $\tilde{A}^1 = \tilde{A}^2 = \tilde{A}^3 = 0$ for all $x'$ [3].

Require that the metric tensor satisfies the Einstein field equations with electromagnetic energy-momentum tensor

$$G_{\mu\nu} = 8\pi T_{\mu\nu} = 8\pi \left[ g^{\sigma\tau}F_{\mu\sigma}F_{\nu\tau} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \right]$$

$$= 8\pi g^{\sigma\tau}[\partial_\mu(g_{\sigma\xi}A^\xi) - \partial_\sigma(g_{\mu\xi}A^\xi)][\partial_\nu(g_{\tau\xi}A^\xi) - \partial_\tau(g_{\nu\xi}A^\xi)] - 2\pi g_{\mu\nu}g^{\alpha\sigma}g^{\beta\tau}[\partial_\alpha(g_{\beta\xi}A^\xi) - \partial_\beta(g_{\alpha\xi}A^\xi)][\partial_\tau(g_{\sigma\xi}A^\xi) - \partial_\sigma(g_{\tau\xi}A^\xi)]$$

(13)

$$- 2\pi g_{\mu\nu}g^{\alpha\sigma}g^{\beta\tau}[\partial_\alpha(g_{\beta\xi}A^\xi) - \partial_\beta(g_{\alpha\xi}A^\xi)][\partial_\tau(g_{\sigma\xi}A^\xi) - \partial_\sigma(g_{\tau\xi}A^\xi)]$$

(14)

A gauge transformation leaves a solution unchanged. Make a gauge transformation and change coordinates so that everywhere the electromagnetic vector potential has zero component one and the other components zero. Independent of what the electromagnetic field is we can thus begin in (13) with $A^0 = 1$ and $A^1 = A^2 = A^3 = 0$ so

$$G_{\mu\nu} = 8\pi g^{\sigma\tau}[\partial_\mu g_{\sigma0} - \partial_\sigma g_{\mu0}][\partial_\nu g_{\tau0} - \partial_\tau g_{\nu0}]$$

$$- 2\pi g_{\mu\nu}g^{\alpha\sigma}g^{\beta\tau}[\partial_\alpha g_{\beta0} - \partial_\beta g_{\alpha0}][\partial_\tau g_{\sigma0} - \partial_\sigma g_{\tau0}]$$

(14)

The flat Minkowski metric is a global solution of (14). There also exist solutions to (14) that are not flat [4].
References


