Finitely Generated Graded Subalgebras of the Lie Algebra of a Smooth Manifold

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Abstract

Let $S(M)$ be the Lie algebra of smooth vector fields on $M$ that smoothly go to zero at infinity if the $n > 1$ dimensional smooth manifold $M$ is not compact. We study the infinite dimensional subalgebras of $S(M)$ generated by $n$ real analytic vector fields that span the tangent space at some point of $M$. It is shown that this is a graded Lie algebra and that two such subalgebras of two manifolds are isomorphic if and only if the manifolds are diffeomorphic.

2000 Mathematics Subject Classification. Primary 17B66; Secondary 17B70

1 Introduction

This article is a revision of De Paepe [1]. Without loss of generality we will take the $n = 2$ case. We will show that $S(M)$ contains two real analytic vector fields $X_1, X_2$ that span the tangent space at some point of $M$ and such that $m(X_1, X_2) \subset S(M)$ is an infinite dimensional graded Lie algebra where $m(X_1, X_2)$ is the Lie algebra generated by $X_1, X_2$. Define $m_k(X_1, X_2)$ to be the span over $\mathbb{R}$ of the set of elements of the form

$$B_k(X_1, X_2) \equiv [X_{j_1}, [X_{j_2}, \cdots [X_{j_{k-1}}, X_{j_k}], \cdots]], j_i = 1, 2 \quad (1)$$

consequently

$$m(X_1, X_2) = \sum_{k=1}^{\infty} m_k(X_1, X_2). \quad (2)$$

Construct $X'_j, m(X'_1, X'_2) \subset S(M')$ for a smooth manifold $M'$ in the same way that $X_j, m(X_1, X_2)$ were instead for $M$. We will show that $m(X_1, X_2)$ and $m(X'_1, X'_2)$ are isomorphic if and only if $M$ and $M'$ are diffeomorphic. We will also show that $B_k(X_1, X_2)$ is a basis element of $m(X_1, X_2)$ if and only if $B_k(E_1, E_2)$ is a basis element of $m(E_1, E_2) \subset S(\mathbb{R}^2)$ where

$$E_j = e^{-x_1^2-x_2^2} \frac{\partial}{\partial x_j}, \quad j = 1, 2. \quad (3)$$

For a smooth atlas of $M$ define a topology on $S(M)$ by the metric

$$\rho(L_1, L_2) = \sum_{q=0}^{\infty} \frac{1}{2^q} \frac{||L_1 - L_2||_q}{1 + ||L_1 - L_2||_q}, \quad L_1, L_2 \in S(M) \quad (4)$$

where $||L_1 - L_2||_q$ is the supremum on $M$ of absolute value of partial derivatives of order less than $k + 1$ of components of $L_1 - L_2$ for some atlas of $M$. Metrics constructed from different atlases are equivalent. Let $\{X_1, X_2\} \subset S(M)$ be analytic vector fields that span the tangent space at some point of $M$ and such that $m(X_1, X_2)$ is infinite dimensional.
2 Isomorphism and diffeomorphism

**Lemma 1** The completion of a non zero ideal $i \subset \mathfrak{m}(X_1, X_2)$ by the $\rho(\cdot, \cdot)$ topology is $S(M)$.

**Proof** Let __

**Theorem 1** If $\mathfrak{m}(X_1, X_2)$ and $\mathfrak{m}(X'_1, X'_2)$ are isomorphic then $M$ and $M'$ are diffeomorphic.

**Proof** Let __

3 Graded subalgebras

Embed $M$ in $\mathbb{R}^4$ so that the resulting point set forms an analytic manifold with no boundary. That this is possible follows from Whitney [2]. Let $Y_q : q = 1, 2, 3, 4$ be the projection of

$$e^{-r^2} \frac{\partial}{\partial x_q}; q = 1, 2, 3, 4, \quad r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

onto the tangent space at each point of $M$. Re-indexing if necessary $Y_1, Y_2$ span the tangent space at some point of $M$. The $Y_1, Y_2$ are analytic vector fields on $M$ and $\mathfrak{m}(Y_1, Y_2)$ is infinite dimensional.

**Theorem 2** $\mathfrak{m}(Y_1, Y_2)$ is a graded Lie algebra.

**Proof** If for example

$$\sum_{k=1}^{N} \sum_{m=1}^{l_k} b_{km} B_{km}(Y_1, Y_2) = 0$$

we must have

$$\sum_{m=1}^{l_N} b_{Nm} B_{Nm}(Y_1, Y_2) = 0$$

since it contains a factor $e^{-Nr^2}$ whereas the other terms do not. __

**Theorem 3** $\mathfrak{m}(X_1, X_2)$ is isomorphic to the graded Lie algebra $\mathfrak{m}(Y_1, Y_2)$.

**Proof** Define $Z_j = X_j + Y_j; j = 1, 2$. Consider

$$\mathcal{B}(Z_1, Z_2) = \sum_{k=1}^{N} \sum_{m=1}^{l_k} a_{km} B_{km}(Z_1, Z_2) = 0.$$  \hspace{1cm} (8)

We can write this equation as

$$\mathcal{B}(Z_1, Z_2) = \mathcal{B}(X_1, X_2) + \mathcal{C}(X, Y) = 0$$  \hspace{1cm} (9)

where each term of $\mathcal{C}(X, Y)$ involves bracket with at least one factor of $Y_1$ or $Y_2$. Each term of $\mathcal{C}(X, Y)$ will then have at least one factor of $e^{-r^2}$. If $X_1$ or $X_2$ had a factor of $e^{-r^2}$ then use instead $e^{-\sqrt{2}r^2}$ in the construction of $Y_1, Y_2$. We have that in order for $\mathcal{B}(Z_1, Z_2)$ to be zero it is necessary that $\mathcal{B}(X_1, X_2)$ is zero. There is then an epimorphism

$$\Lambda : \mathfrak{m}(Z_1, Z_2) \to \mathfrak{m}(X_1, X_2)$$  \hspace{1cm} (10)

which takes $Z_1$ to $X_1$ and $Z_2$ to $X_2$. Show $\Lambda$ is continuous. Let $\mathfrak{k}$ be the kernel of the map $\Lambda$. If $\mathfrak{k} \neq 0$ then using lemma 1 we would have the completion of $\mathfrak{k}$ by the $\rho$ topology would be $S(M)$ which is a contradiction. __
Since \( m(Y_1, Y_2) \) is a graded Lie algebra we have that \( m(X_1, X_2) \) is a graded Lie algebra.

Let \( \bigcup_{q=1}^{\infty} \{ v_{kq}, v_k : k = 1, 2, \ldots, l \} \) be a set of vectors. Let there be a norm topology on the span over \( \mathbb{R} \) of this set. Require also that as \( q \to \infty, v_{kq} \to v_k. \)

**Lemma 2** If \( \{ v_1, v_2, \ldots, v_l \} \) is a linear independent set then there is a \( N \) such that for all \( q > N, \{ v_{1q}, v_{2q}, \ldots, v_{lq} \} \) is a linearly independent set.

**Proof** Assume there are \( n \) with \( n \to \infty \) so that for each \( n \) the set of vectors \( \{ v_{kn} : k = 1, 2, \ldots, l \} \) are dependent. There are then \( a_{kn} \) so that

\[
\sum_{k=1}^{l} a_{kn}^2 = 1, \quad \sum_{k=1}^{l} a_{kn} v_{kn} = 0.
\]  

(11)

Let \( a_k \) be the limit of a subsequence of the \( a_{kn} \). Taking the limit in Eqs. 11 we have

\[
\sum_{k=1}^{l} a_k^2 = 1, \quad \sum_{k=1}^{l} a_k v_k = 0
\]  

(12)

which is a contradiction. \( \Box \)

**Theorem 4** \( B_{km}(X_1, X_2) \) is a basis element of \( m \) iff \( B_{km}(E_1, E_2) \) is a basis element of \( m(E_1, E_2) \).

**Proof** Let \( U_M \) and \( U_{\mathbb{R}^2} \) open subsets of \( M \) and \( \mathbb{R}^2 \) respectively so that there is a diffeomorphism \( \phi \) of these sets. Let \( G_j ; j = 1, 2 \) be the image of the mapping by \( \phi \) of \( X_1, X_2 \) restricted to \( U_M \). Let

\[
\{ B_{km}(G_1, G_2) : m = 1, \ldots, l_k \}
\]  

be a basis set of \( m_k(G_1, G_2) \). Since \( X_1, X_2 \) are analytic a basis set of \( m_k(X_1, X_2) \) will be

\[
\{ B_{km}(X_1, X_2) : m = 1, \ldots, l_k \}
\]  

(14)

Using lemma 2 there are analytic \( F_1, F_2 \subseteq S(\mathbb{R}^2) \) each with restrictions to \( U_{\mathbb{R}^2} \) that are close enough in the \( \| \cdot \|_0 \) as in (4) but restricted to \( U_{\mathbb{R}^2} \) topology to \( G_1, G_2 \) so that the basis set of \( m_k(F_1, F_2) \) includes the set

\[
\{ B_{km}(F_1, F_2) : m = 1, \ldots, l_k \}
\]  

(15)

Now use theorem 3. Interchange \( X_1, X_2 \) and \( E_1, E_2 \) in the argument. \( \Box \)

Since the dimension of \( m_k(E_1, E_2) \) grows polynomially with \( k \) the dimension of \( m_k(X_1, X_2) \) grows polynomially with \( k \) hence \( m(X_1, X_2) \) is not a free Lie algebra.

**Theorem 5** \( m(X_1, X_2) \) and \( m(X'_1, X'_2) \) are isomorphic if and only if \( M \) and \( M' \) are diffeomorphic.

**Proof** In theorem 1 we showed that if \( m(X_1, X_2) \) and \( m(X'_1, X'_2) \) are isomorphic then \( M \) and \( M' \) are diffeomorphic.

Let \( M' \) and \( M \) be diffeomorphic. Embed these into \( \mathbb{R}^4 \) so that the resulting point sets are two analytic manifolds. Let \( \sigma \) be an analytic diffeomorphism of the embedded manifolds. The vector fields \( \sigma_*(X_1) \) and \( \sigma_*(X_2) \) are then analytic and \( m(X_1, X_2) \) and \( m(\sigma_*(X_1), \sigma_*(X_2)) \) will be isomorphic. Using theorem 3 there is an isomorphism of \( m(\sigma_*(X_1), \sigma_*(X_2)) \) and \( m(X'_1, X'_2). \) \( \Box \)

**References**
